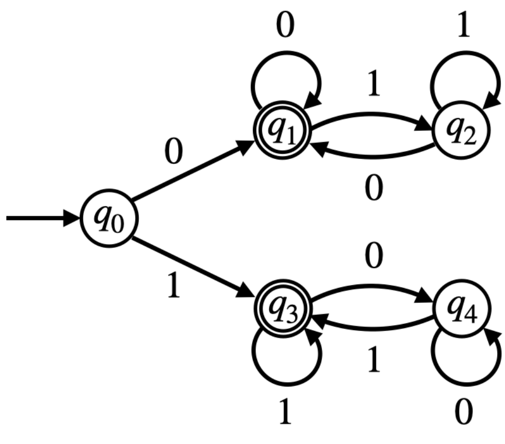
Maths for Computing
Assignment 6 Solutions

1. *(5 marks)* Construct a DFA for $L = \{w \mid w$ contains equal number of occurrences of 01 and 10 as substring of $w\}$ over $\{0,1\}$. For instance, $010 \in L$, $0110 \in L$, $0111 \notin L$, $10110 \notin L$. Justify with a casual proof why the constructed DFA is deciding L.

Solution: If a string, say w, starts and ends with a 0, it must contain equal number of occurrences of 01 and 10. The intuition is that w can be divided into an odd number of substrings $w_1w_2\ldots w_{2k+1}$ such that odd indexed strings are made of 0s and even indexed strings are made of 1s. In such a division, both, number of occurrences 01s and 10s will be equal to k. Similarly, we can observe that if a string starts and ends with a 1, it must contain equal number of occurrences of 01 and 10.

Now, if a string starts with a 0 and ends with a 1, using the similar division we can observe that number of 01s will be one more than the number of 10s. And we can say the vice verse for strings starting from 1 and ending with a 0.

Therefore, we can simply construct a DFA that checks whether a string starts and ends with the same symbol. The below DFA does that:



2. *(5 marks)* Prove or disprove that $L = \{0^n \mid n$ is a perfect square$\}$ over $\{0,1\}$ is a regular language.

Solution: Suppose L is regular and has a DFA of n states. Let us take $w = 0^{n^2}$. Clearly, $w \in L$ as its length is a perfect square and $|w| \geq n$.

We show now that it is impossible to split w in 3 strings, $w = xyz$, so that it satisfies all three conditions of pumping lemma.

1. If y contains nothing, it will violate condition 1.

2. Suppose $y = 0^l$. If $l > n$, then it will violate condition 2 as $|xy| > n$. Now suppose $l \leq n$ and consider the string xy^2z. Length of xy^2z is $n^2 + l$, which is not a perfect square as $n^2 + l$ is lying strictly between two perfect squares n^2 and $(n+1)^2$,

$$n^2 < n^2 + l < (n+1)^2 = n^2 + 2n + 1.$$

Hence, $xy^2z \notin L$ and this violates the 3rd condition.

3. *(5 marks)* Construct a CFG for $L = \{a^i b^j c^k \mid i \neq j \text{ or } j \neq k, \text{ where } i, j, k \in \mathbb{Z}^+\}$ over $\{a, b, c\}$.

Solution: Variables $= \{S, X, Y, A, B, C\}$, terminals $= \{a, b, c\}$, and the start symbol is S. Productions are given below:

$$\begin{aligned}
S &\to aXbC \mid AbYc \\
X &\to aXb \,|\, A \,|\, B \\
Y &\to bYc \,|\, B \,|\, C \\
A &\to aA \,|\, a \\
B &\to bB \,|\, b \\
C &\to cC \,|\, c
\end{aligned}$$

No justification is required, if your grammar is correct.